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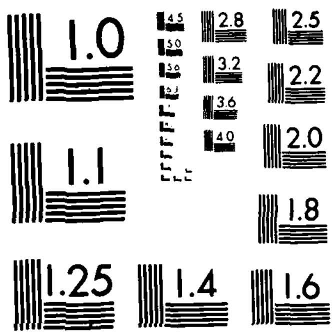

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LIMIT THEOREMS FOR THE EIGENVALUES OF
PRODUCT OF TWO RANDOM MATRICES

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1. INTRODUCTION

The distributions of the eigenvalues or functions of the eigenvalues of random matrices are very useful in testing various hypotheses in multivariate statistical analysis. These distributions are useful in nuclear physics also since the behaviour of the energy levels at high excitation levels in nuclear physics may be explained by considering the distributions of the eigenvalues of certain random matrices. In the area of multivariate statistical analysis, the asymptotic distribution theory is essentially restricted to the case when the sample size tends to infinity holding the number of variables fixed. But, many situations arise when the experimenter is confronted with the problem of drawing inference from the data when the number of variables is very large. Wigner [7] considered the problem of deriving the distributions of the eigenvalues of the "Gaussian matrix" when the number of variables tends to infinity; Jonsson [4,5]. Waçhter [6] and others have investigated the distributions of the eigenvalues of the sample covariance matrix when the number of variables tends to infinity. In this paper, we will show that the spectral distributions of a sequence of the products of the random matrices will tend to the distribution function in the limit as the number of variables tend to infinity. An application of this result in deriving the distributions of eigenvalues of the multivariate F matrix when the number of variables tends to infinity will be discussed in a subsequent paper.

2. PRELIMINARIES

Let A_p be a p×p matrix with real eigenvalues $\ell_1 \leq \ell_2 \leq \ldots \leq \ell_p$. Then, we define a distribution function as

$$F_p(x) = \frac{1}{p} \#\{i: \ell_i \leq x\},$$

where #{ } denotes the number of elements of the set { }. We call this function the spectral distribution function of A_p.

We are interested here in a sequence $\{A_p\}$ of random matrices, each A_p has only real eigenvalues. If the spectral distribution $F_p(x)$ of A_p tends to a nonrandom distribution function $F(x)$ as $p \to \infty$, in some sense, then we say that the sequence $\{A_p\}$ has a limit spectral distribution (in the given sense) $F(x)$.

Jonsson [4,5] proved that a sequence of Wishart matrices has a limit spectral distribution. Wachter [6] got more general results, but he still considered matrices of Wishart type as was done by Grenander and Silverstein [3].

In this article, we consider a different type of random matrices, namely products of certain random matrices.

Let X_{ij}, i, j = 1,2,... be distributed independently and identically as normal with mean zero and variance one. Also, let

$$W_p = X_p X_p^T$$

be a Wishart matrix where

$$X_p = (X_{ij},\ 1 \leq i \leq p,\ 1 \leq j \leq m).$$

Then W_p is known to be the Wishart matrix with m degrees of freedom.

For each $p \geq 1$, let $T_p = (t_{ij}^{<p>},\ 1 \leq i \leq p,\ 1 \leq j \leq p)$ be a matrix of random variables. We suppose $t_{ij}^{<p>} = t_{ji}^{<p>}$, and $T_p \geq 0$, for any $i, j = 1,2,\ldots,p$, $p = 1,2,\ldots$

Our main result is the following theorem.

Theorem. Suppose

(1) $\{X_{ij}\}$ and T_p's are independent

(2) the limit $\lim_{m,p\to\infty} \frac{p}{m} = y$ exists and is finite

(3) if $G_p(x)$ is the spectral distribution of T_p, for each fixed k, $E\int x^k d\, G_p(x)$ are bounded as $p \to \infty$.

(4) $\lim_{p\to\infty} \int x^k dG_p(x) = H_k$ exists, for $k = 1,2,\ldots$, in $L^2(P)$,

and $\sum_{k=1}^{\infty} H_{2k}^{-\frac{1}{2k}} = +\infty$ (Carleman's condition, see Feller[2])

Then, the sequence $\{\frac{1}{m} W_p T_p\}$ has a limit spectral distribution in probability, i.e. if F_p is the spectral distribution of $\{\frac{1}{m} W_p T_p\}$ then there is a distribution function F such that $F_p(x) \to F(x)$ in pr. as $p \to \infty$ for any x. $F(x)$ is nonrandom.

If $F_p(x)$ is the spectral distribution of $\frac{1}{m} W_p T_p$, and $M_k = \int x^k dF_p(x)$ is the k-th moment of $F_p(x)$, we shall prove the theorem by proving

(5) $E_k = \lim_{p\to\infty} EM_k$ exists for $k = 1,2,3,\ldots,$

(6) $\mathrm{Var}\, M_k \to 0$ as $p\to\infty$, for $k = 1,2,\ldots,$

(7) $\Sigma E_{2k}^{-\frac{1}{2k}} = +\infty$.

The main difficulty is to prove (5). In order to prove (5), we need to develop a theory on a special kind of multi-graphs and we call them Q-graphs. Thus our work involves combinatorial problems.

We will use the following lemma.

Lemma A. $E\left|\int x^{k_1} dG_p(x) \ldots \int x^{k_h} dG_p(x) - H_{k_1} \ldots H_{k_h}\right| \to 0$ as $p \to \infty$, for any fixed positive integers $k_1, \ldots, k_h$.

Proof. We prove by induction on h. For h = 1, this is a direct consequence of condition (4). Suppose Lemma A is true for h - 1. Then

$$E_p^{(h)} = E\left|\int x^{k_1} dG_p(x) \ldots \int x^{k_h} dG_p(x) - H_{k_1} \ldots H_{k_h}\right|$$

$$\leq E\left|\int x^{k_1} dG_p(x) \ldots \int x^{k_{h-1}} dG_p(x)\right| \left|\int x^{k_h} dG_p(x) - H_{k_h}\right|$$

$$+ E\left|\int x^{k_1} dG_p(x) \ldots \int x^{k_{h-1}} dG_p(x) - H_{k_1} \ldots H_{k_{h-1}}\right| H_{k_h}$$

$$\leq E^{\frac{1}{2}}\left|\int x^{k_1} dG_p(x) \ldots \int x^{k_{h-1}} dG_p(x)\right|^2 \cdot E^{\frac{1}{2}}\left|\int x^{k_h} dG_p(x) - H_{k_h}\right|^2 + E_p^{(h-1)} H_{k_h}.$$

So, we have only to prove $E\left|\int x^{k_1} dG_p(x) \ldots \int x^{k_{h-1}} dG_p(x)\right|^2$ is bounded.

Let $k = \max(k_1, \ldots, k_{h-1})$. For some $a \geq 2$, by Hölder inequality,

$$\left|\int x^{k_1} dG_p(x) \ldots \int x^{k_{n-1}} dG_p(x)\right|^2 \leq \left|\int x^{k} dG_p(x)\right|^a \leq \int x \quad dG_p(x).$$

But if $k_1 \geq ka$ is any integer, by condition (3),

$$E\int x^{ka} dG_p(x) = E\int_{x<1} x^{ka} dG_p(x) + E\int_{x\geq 1} x^{ka} dG_p(x)$$

$$\leq 1 + E\int_{x\geq 1} x^{K_1} dG_p(x) \leq 1 + E\int_{x\geq 1} x^{K_1} dG_p(x)\,.$$

which is bounded.

3. SOME RESULTS ON GRAPH THEORY

We first prove some results on graph theory.

Let V, E be two finite sets. Suppose there is a function $g: E \to V \times V$. Then (V,E,g) is called a multidigraph. If $x \in V$, $y \in V$, (x,y) will denote one of those edges in E whose g image is (x,y), sometimes we write xy instead of (x,y).

If $v \in V$ occurs in the list $\{g(e) = (g_1(e), g_2(e)),\ e \in E\}$ as $g_1(e)$ or $g_2(e)$ just d times, then we say that the degree of v is d.

<u>Definition</u>. Let (V,E,g) be a multidigraph. If it satisfies the following conditions, then we say that it is a Q-graph:

1°. Each vertex (i.e. element of V) has degree 2.

2°. V is divided into disjoint classes such that the graph is class-connected, i.e. for any two classes A and B there are classes $A = A_0, A_1, \ldots, A_r = B$ and edges (x_i, y_i) (i.e. element of E) with $x_i \in A_{i-1}$, $y_i \in A_i$, $i = 1, \ldots, r$.

For Q-graphs we have the following results.

<u>Lemma 1</u>. A Q-graph G with k vertices and w classes consists of at most $k-w+1$ cycles (we see loops as cycles). G consists of just $k-w+1$ cycles if and only if

1°. Each cycle meets each class in at most one vertex.

2°. There can be no such sequences as

$A_1, C_1, A_2, C_2, \ldots, A_r, C_r, A_1$

where A_i's are different classes, C_i's are different cycles and C_i meets A_i and A_{i+1} for $i = 1, 2, \ldots, r$ $(A_{r+1} = A_1)$.

Proof. It is evident that a Q-graph consists of disjoint cycles.

Let G be a Q-graph with k vertices and w classes. Suppose G has maximum number of cycles. We prove that conditions 1^o and 2^o are fulfilled.

Suppose that cycle C of G meets class A in x and y, $x \neq y$ are two vertices. We replace C by two loops (x,x) and (y,y) in case C has only two vertices. Otherwise, suppose

$$C = x_1 x_2 x_3 \dots x_r x_1$$

in which $x_i = x$, $x_j = y$. Then we replace C by loop (x,x) and cycle

$$C' = x_1 \dots x_{i-1} x_{i+1} \dots x_r x_1.$$

The resulting graph is still a Q-graph with k vertices and w classes, but the number of cycles has increased. This contradicts that the number of cycles of the graph G is maximal.

Suppose 2^o is not satisfied, and there is a sequence

$$A_1, C_1, A_2, C_2, \dots, A_r, C_r, A_1$$

of different classes A_i's and different cycles C_i's, such that C_i meets A_i and $A_{i+1}(A_{r+1}=A_1)$. We replace C_r by loops constructed from all vertices on C_r which belong to A_1 and the cycle obtained from C_r by deleting all these vertices. The resulting graph is also a Q-graph with k vertices and w classes, but with more cycles.

Thus 1^o and 2^o are satisfied.

In the above proof we see that any Q-graph with k vertices and w classes can be replaced by a Q-graph with the same k and w, with not less cycles and the latter Q-graph satisfies 1^o and 2^o.

In the following we prove that any Q-graph with k vertices and w classes satisfying 1^{o} and 2^{o} must have k-w+1 cycles.

Let C_1 and C_2 be two arbitrary cycles, passing through some class A simultaneously, and both contain vertices outside of A, $C_1 = \ldots x_1 y_1 z_1 \ldots$, $C_2 = \ldots x_2 y_2 z_2 \ldots$, $y_1, y_2 \in A$. Then we replace C_1 and C_2 by C_1' and C_2', and C_1' is the loop $y_1 y_1$, $C_2' = \ldots x_1 y_2 z_2 \ldots x_2 z_1 \ldots$

Because class-connectivity is preserved, this procedure can be continued until there remains only one cycle C which is not a loop. C must meet every class. We note that in this course k,w and the number of cycles do not change. C has w vertices, the remaining vertices constitute k-w loops. So, there are k-w+1 cycles.

Let G be a multigraph (V,E,g). Let F be a partition of V into disjoint subsets, i.e. F is a set of subsets of V, these subsets are mutually disjoint and their union is V. Let $f: V \to F$ be the mapping such that f(v) is the subset in F which contains v, for any $v \in V$. If we use $f \times f$ to denote the mapping $V \times V \to F \times F$ with $f \times f: (v_1, v_2) \mapsto (f(v_1), f(v_2))$, then evidently $\tilde{G} = (F,E,(f \times f)og)$ is also a multigraph. This multigraph is said to be obtained from G by identification according to F, and a subset in F with more than one vertex is called an identified vertex.

In the following, by an arc of $\tilde{G}$ we refer to a finite sequence of edges of $\tilde{G}$ $x_1x_2, x_2x_3, \ldots x_{r-1}x_r$ such that only x_1 and x_r are identified vertices, i.e. only they are subsets in F with more than one element.

Lemma 2. Let $G = (V,E,g)$ be a multigraph with k vertices and w classes, and $\tilde{G} = (F,E,(f \times f)og)$ be the multigraph obtained from G by identification according to a partition F of V. Let ξ be the number of arcs in $\tilde{G}$ and η be the number of free cycles (those cycles of $\tilde{G}$ on which there are no identified vertices). Then $\xi/2 + \eta \leq k-w$.

Proof. Let $\Gamma_0, \Gamma_1, \ldots, \Gamma_d$ be the cycles of G. They are so arranged that for any i, Γ_i passes through a class which contains vertices of some of the cycles $\Gamma, \ldots, \Gamma_{i-1}$. Cycles $\Gamma_o, \ldots, \Gamma_i$ constitute a subgraph G_i of G. If we identify vertices of G_i in such a way that 2 vertices are identified if and only they belong to the same subset in F, we get a multigraph $\tilde{G}_i$, $\tilde{G}_d = \tilde{G}$.

Let a_i be the number of arcs in $\tilde{G}_i$ (with respect to the identified multigraph $\tilde{G}_i$). Evidently a_i increases with i, and $a_d = \xi$, $a_o = 0$. Let $\eta_i = 1$ or 0 according as Γ_i is free or not (with respect to $\tilde{G}$), and ζ_i be the number of vertices of Γ_i which belong to classes containing vertices of $\Gamma_0 \ldots \Gamma_{i-1}$.

We assert $\frac{1}{2}(a_i - a_{i-1}) + \eta_i \leq \zeta_i$, $i = 1,2,\ldots,r$.

If $\eta_i = 1$, then $a_i - a_{i-1} = 0$, but $\zeta_i > 0$, so the inequality holds.

Now assume $\eta_i = 0$. Take a class A of G, which contains some vertices of $\Gamma_0,\ldots,\Gamma_{i-1}$ and some vertices of Γ_i. Let the common vertices of A and Γ_i be $x_1,\ldots,x_e$ among which $x_1,x_2,\ldots,x_b$ take part into identification with vertir of $\Gamma_0 \ldots \Gamma_{i-1}$ or that of Γ_i. There are five ways to id ify:

(1) A single x_c is identified with an identifi vertex y of $\tilde{G}_{i-1}$,

(2) A single x_c is identified with an unidentified vertex y of $\tilde{G}_{i-1}$,

(3) Several x_c's are identified with an identified vertex y of $\tilde{G}_{i-1}$ together,

(4) Several x_c's are identified with an unidentified vertex y of $\tilde{G}_{i-1}$ together.

(5) Several x_c's are identified together, without vertices of G_{i-1} taking part into this identification.

and the increment of number of arcs by the above five different ways of identification are the following, respectively:

(1) one,

(2) two,

(3) number of these X_c's

(4) number of these X_c's plus one,

(5) number of these X_c's.

In any case, the increment of the number of arcs does not exceed twice the number of vertices of Γ_i in A taking part into these identifications. Summing over all such classes A, we get

$$\frac{1}{2}(a_i - a_{i-1}) + \eta_i \le \zeta_i, \quad i = 1,\dots,r,$$

because $\eta_i = 0$. Thus

$$\frac{1}{2}\zeta + \eta \le \sum_{i=1}^{r} \zeta_i .$$

We now prove $\sum_1^r \zeta_i \le k-w$, by proving that there is a Q-graph with k vertices and w classes but with more than $\sum_1^r \zeta_i$ cycles, so, $\Sigma\, \zeta_i < k-w+1$.

Begin from Γ_1. Γ_1 has ζ_1 vertices belonging to classes in which there are vertices of Γ_0. We let x_1, be one of these vertices. If there is only one such vertex, then we let it remain there untouched. Otherwise, if $\Gamma_1 = \dots y\ x_1\ z \dots$, then we replace Γ_1 by the loop (x_1,x_1) and regard Γ_1 the cycle ...yz... We continue this process until there is only one vertex of Γ_1 which is in some class with some vertices of Γ_0 together. Then we get, from $\Gamma_1 (\zeta_1 - 1)$ loops and a remaining cycle, totally ζ_1 cycles. In the same way, we get, from Γ_2, ζ_2 cycles, ... At last, we get a Q-graph with k vertices and w classes and $1 + \zeta_1 + \dots + \zeta_r$ cycles, here 1 is for the cycle Γ_0. Therefore $\zeta_1 + \dots + \zeta_r \le k-w$. And the lemma is proved.

Let $A = \{A_1,\dots,A_w\}$ be a partition of $\{1,\dots,k\}$ into disjoint classes, and $B = \{B_1,\dots,B_v\}$ be a partition of $\{1,2,\dots,2k\}$ into disjoint classes. B will be called a partition subject to A if every B class is included in some set of the form $A_b^* = (2A_b - 1) \cup (2A_b)$. B is called even if every class of it contains just even number of elements. If every class of B has just 2 numbers then B will be called a pairing.

Lemma 3. Let $A = \{A_1, \ldots, A_w\}$, $B = \{B_1, \ldots, B_v\}$ be partitions of $\{1,2,\ldots,k\}$ and $\{1,2,\ldots,2k\}$ respectively, B be subject to A and even. Suppose at least one B class has more than 2 elements. Then,

$$S = \sum{}' t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1} = o(p^{k-w+1}), \quad \text{in } L^2. \; *$$

Here $\sum'$ means the summation is taken over all $(i_1, i_2, \ldots i_{2k})$ for which $1 \le i_1 \le p, \ldots, 1 \le i_{2k} \le p$, and if α, β belong to the same B class, $i_\alpha = i_\beta$.

Proof. It is evident that we can find a pairing D of the set $\{1,2,\ldots,2k\}$ subject to B, i.e. each pair is a subset of some B-class.

If we regard $i_2 i_3, i_4 i_5, \ldots, i_{2k}\, i_1$ as edges and pair $\{i_a, i_b\}$ as vertics if a, b are a pair in D, we get a multigraph G. If we classify the vertices of G according to A, i.e. two vertices $\{i_a, i_b\}$, $\{i_c, i_d\}$ belong to the same class if and only if these four integers a,b,c,d all belong to an $A_q^* = (2A_q - 1) \cup (2A_q)$. Thus we get a Q-graph, its class-connectivity is easy to see, and it is evident that each vertex has degree 2.

We can further identify the vertices of G into another multigraph $\tilde{G}$ according to the partition B just as in Lemma 2, two vertices are identified if and only if the pairs defining

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*In the following, we will write t_{ij} instead of $t_{ij}^{<p>}$.

them have indices included in the same B class.

Because a Q-graph consists of disjoint cycles, we can write the sum S as

$$S = \sum{}'' C_1 C_2 \dots C_b,$$

here C_a's are "cycles" of the form: $C_a = t_{j_1 j_2} t_{j_2 j_3} \dots t_{j_c j_1}$, corresponding to cycle $j_1 j_2 j_3 \dots j_c j_1$ of G. $\sum''$ means we have to identify further the indices whose subscripts belong to the same B class.

For those cycles which are free, i.e. on which all vertices need not further identifications, the summation can be carried out:

$$\sum_{j_1,\dots,j_\ell} t_{j_1 j_2} t_{j_2 j_3} \dots t_{j_\ell j_1} = \mathrm{tr}\, T_p^\ell = \int x^\ell dG_p(x) \cdot p$$

The remaining cycles are not free, they have indices identified with other indices. But on these cycles there are two kinds of indices, free and not free. Free means it occurs just twice.

Carrying out the summation with respect to free indices, by the definition of multiplication of matrices,

$$S = \sum_{a,b} t_{a_1 b_1}^{(n_1)} \dots t_{a_\xi b_\xi}^{(n_\xi)} \mathrm{tr}\, T_p^{k_1}\, \mathrm{tr}\, T_p^{k_2} \dots \mathrm{tr}\, T_p^{k_\eta}, \;*$$

here ξ, η have the same meanings as in Lemma 2. Notice that each of the indices $a_1, b_1 \dots, a_\xi, b_\xi$ occurs at least four times in this list.

In general, let the general term of a sum of the form

* $t_{ij}^{(n)}$ denotes an entry of the matrix T_p^n.

$$R = \sum_{a_1,\ldots} f_1(a_1)\ldots f_r(a_r) g_1\left(a_{i_1}, a_{j_1}\right) g_2\left(a_{i_2}, a_{j_2}\right)\cdots g_s\left(a_{i_s}, a_{j_s}\right)$$

be a product, and each factor of it depends at most on two indices and each index a_i occurs at least four times, where the summation is over $a_1,\ldots,a_r$, $a_{i_1},\ldots,a_{i_s}$, $a_{j_1},\ldots,a_{j_s}$. Then,

$$R^2 \le \sum_{a_1} f_1^2(a_1)\ldots \sum_{a_r} f_r^2(a_r) \sum_{a_{i_1},a_{j_1}} g_1^2\left(a_{i_1}, a_{j_1}\right)\cdots \sum_{a_{i_s},a_{j_s}} g_s^2\left(a_{i_s}, a_{j_s}\right)$$

In fact, by Schwarz inequality

$$R^2 \le \sum_{a_1} f_1^2(a_1) \sum_{a_1}\left(\sum_{a_2,\ldots} F(a_1,a_2,\ldots)\right)^2.$$

The sum inside the bracket has the same property as R if we regard a_1 as constant. The inequality is thus proved by induction. The case $r = 0$ can be proved similarly. So,

$$S^2 \le \sum_{a_1,b_1}\left[t_{a_1b_1}^{(n_1)}\right]^2\cdots\sum_{a_\xi,b_\xi}\left[t_{a_\xi b_\xi}^{(n_\xi)}\right]^2\cdot\left(\operatorname{tr} T_p^{k_1}\right)^2\ldots\left(\operatorname{tr} T_p^{k_2}\right)^2.$$

But, because of symmetry, $\sum_{a,b}[t_{ab}^{(n)}]^2 = \operatorname{tr} T_p^{2n}$, thus

$$S^2 \le \left(\frac{1}{p}\operatorname{tr} T_p^{2n_1}\right)\ldots\left(\frac{1}{p}\operatorname{tr} T_p^{2n_\xi}\right)\left(\frac{1}{p}\operatorname{tr} T_p^{k_1}\right)\ldots\left(\frac{1}{p}\operatorname{tr} T_p^{k_\eta}\right)^2\cdot p^{\xi+2\eta}.$$

So, by Lemma A since $\frac{1}{p}\operatorname{tr} T_p^q = \int \chi^q dG_p(X)$,

$$S = O\left(p^{\frac{\xi}{2}+\eta}\right) = o\left(p^{k-w+1}\right) \quad \text{in } L^2$$

by Lemma 2.

4. PROOF OF EXISTENCE OF $E(M_k)$ WHEN $p \to \infty$

Let $F_p(X)$ be the spectral distribution of the matrix $\frac{1}{m}W_pT_p$, i.e., $F_p(X) = \frac{1}{p}\#\{i : \ell_i \le X\}$, here $\ell_1 \le \ldots \le \ell_p$ are eigenvalues of $\frac{1}{m}W_pT_p$. Note that because $W_pT_p = W_p^{\frac{1}{2}}W_p^{\frac{1}{2}}T_p$ and $W_p^{\frac{1}{2}}T_pW_p^{\frac{1}{2}}$ have the same eigenvalues and $W_p^{\frac{1}{2}}T_pW_p^{\frac{1}{2}}$ is symmetric, $\ell_1,\ldots,\ell_p$ are all real.

Let

$$M_k = \int x^k dF_p(x), \quad k = 1,2,\ldots .$$

In this section we will prove that $EM_k \to E_k$ as $p \to \infty$ and

$$\sum E_{2k}^{-\frac{1}{2k}} = \infty.$$

We have

$$M_k = \int x^k dF_p(x) = \frac{1}{p}\sum_{i=1}^{p} \ell_i^k = \frac{1}{p}\operatorname{tr}\left(\frac{1}{m}W_pT_p\right)^k$$

$$= \frac{1}{pm^k}\sum w_{i_1i_2}t_{i_2i_3}w_{i_3i_4}t_{i_4i_5}\cdots w_{i_{2k-1}i_{2k}}t_{i_{2k}i_1},$$

here the sum is taken with respect to each index $i_1, i_2, \ldots, i_{2k}$ running from 1 to p. Remember

$$w_{ii'} = \sum_{j=1}^{m} X_{ij}X_{i'j},$$

So

$$M_k = \frac{1}{pm^k}\sum X_{i_1j_1}X_{i_2j_1}X_{i_3j_2}X_{i_4j_2}\cdots X_{i_{2k-1}j_k}X_{i_{2k}j_k}t_{i_2i_3}t_{i_4i_5}\cdots t_{i_{2k}i_1}$$

here $j_1,\ldots,j_k$ run from 1 to m.

Noticing X_{ij} and t_{ij} are independent, we have

$$EM_k = \frac{1}{pm^k}\sum E\prod_{q=1}^{k}\left(X_{i_{2q-1}j_q}X_{i_{2q}j_q}\right)\cdot E\,t_{i_2i_3}t_{i_4i_5}\cdots t_{i_{2k}i_1}.$$

Since for different (i,j), X_{ij}'s are independent, we would collect the factors $X_{i_{2q-1}j_q} X_{i_{2q}j_q}$ together for equal j_q's, and split such factors for different j_q's. Thus for each X-product we have a partition $A = \{A_1, \ldots, A_w\}$ of the set $\{1,2,\ldots,k\}$ such that two integers q and q' belong to the same A class iff $j_q = j_{q'}$. The whole sum is then split into a sum of sums, each of the latter has the same partition A. Thus,

$$E\,M_k = \frac{1}{pm^k} \sum_A \sum_{(j)|A} \sum_{(i)} E \prod_{q=1}^{k} \left(X_{i_{2q-1}j_q} X_{i_{2q}j_q} \right) \cdot E\, t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1},$$

here $\sum_A$ means summation over all possible partitions of $\{1,2,\ldots,k\}$, $(j)|A$ means $j_q = j_{q'}$ iff q and q' belong to a common A class.

Given $A = \{A_1, \ldots, A_w\}$, a partition of $\{1,2,\ldots,k\}$, given vector $(j) = (j_1, \ldots, j_k)$ such that $j_a = j_b$ iff a,b belong to the same A class, we denote the different w values of $j_1, \ldots, j_k$ by $r_1, \ldots, r_w$, and understand that $r_a = j_b$ if $b \in A_a$. Then the inner sum of $E\,M_k$ is

$$S_{A,(r)} = \sum_{(i)} \prod_{a=1}^{w} E \prod_{b \in A_a} \left(X_{i_{2b-1}r_a} X_{i_{2b}r_a} \right) E \left(t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1} \right).$$

Given a partition A of $\{1,2,\ldots,k\}$ we can correspond a partition A^* of $\{1,2,\ldots,2k\}$ in such a way: if $A = \{A_1, \ldots, A_w\}$, then $A^* = \{A_1^*, \ldots, A_w^*\}$, here $A_a^* = (2A_a - 1) \cup (2A_a)$, or equivalently, $n \in A_a^*$ iff $[\frac{n+1}{2}] \in A_a$. With the aid of A^*, we can write

$$S_{A,(r)} = \sum_{(i)} \prod_{a=1}^{w} E\left(\prod_{b\in A_a^*} X_{i_b r_a}\right) \cdot E\, t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1}.$$

In order that a term in this sum does not vanish, it is necessary that each $E \prod_{b\in A_a^*} X_{i_b r_a} \neq 0$. And if this holds, A_a^* must be further partitioned into classes, each class contains even number of elements and $i_b = i_{b'}$ iff b,b' belong to the same class. As a whole, we must have a partition B of $\{1,2,\ldots,2k\}$, it is even, a refinement of A^*, and if b,b' belong to the same A^* class, then $i_b = i_{b'}$ iff b and b' belong to the same B class, in order that $\prod_{a=1}^{w} E \prod_{b\in A_a^*} X_{i_b r_a} \neq 0$. In this case.

$$\prod_{a=1}^{w} E \prod_{b\in A_a^*} X_{i_b r_a} = K(A^*,B)$$

depends only on A^*,B and is independent of special values of (i).

Thus

$$S_{A,(r)} = \sum_{\substack{B\geq A^* \\ B \text{ even}}} \sum_{(i)|A^*,B} K(A^*,B)\, E\, t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1}.$$

Here $B\geq A^*$ means partition B of $\{1,2,\ldots,2k\}$ is a refinement of A^*, $(i)|A^*,B$ means if b,b' both belong to a same A^* class then $i_b = i_{b'}$ iff b and b' belong to the same B class. We see that $S_{A,(r)}$ is independent of (r).

Suppose B is a pairing. Then we can define a Q-graph $G(A^*,B)$. The edges are $i_2 i_3, i_4 i_5, \ldots, i_{2k} i_1$. The vertices are pairs $\{i_a, i_b\}$ where $\{a,b\}$ is a B-class. The classes of vertices are determined by A^*.

Case 1. B is a pairing and $G(A^*,B)$ has less than $k-w+1$ cycles. We consider the sum

$$S_{A*,B} = \sum_{(i)|A*,B} E\, t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1}.$$

Because $G(A^*,B)$ is a Q-graph, $i_2 i_3, i_4 i_5, \ldots, i_{2k} i_1$ constitute $v < k-w+1$ cycles. Thus

$$S_{A*,B} = \sum_{j_1,\ldots,j_k}{}' E\, C_1 C_2 \ldots C_v,$$

here C_a's are "cycles", $C_a = t_{j_{a_1} j_{a_2}} t_{j_{a_2} j_{a_3}} \cdots t_{j_{a_c} j_{a_1}}$, $\sum'_{j_1,\ldots,j_k}$ means the sum is taken with respect to k identified indices $j_1,\ldots,j_k$ varying from 1 to p but when two of them belong to the same class of vertices determined by A*, they must keep different mutually.

Thus $S_{A*,B}$ can be expressed as

$$S_{A*,B} = \sum_{I_1,\ldots,I_u} E\, C_1 C_2 \ldots C_v,$$

here $I_1,\ldots,I_u$ are inequalities of the form $j_a \neq j_b$ between two indices belonging to the same class of vertices. By inclusion-exclusion principle,

$$S_{A*,B} = \\ = \sum_{j_1,\ldots,j_k} - \left(\sum_a \sum_{\tilde{I}_a} - \sum_{a<b} \sum_{\tilde{I}_a \tilde{I}_b} + \sum_{a<b<c} \sum_{\tilde{I}_a, \tilde{I}_b, \tilde{I}_c} - \cdots \right)$$

here $\tilde{I}_a$ denotes the negation of I_a. By Lemma 3 the terms in the bracket are all $o(p^{k-w+1})$, and by Lemma A

$$\sum_{j_1\ldots,j_k}{}' = E\, \mathrm{tr}\, T_p^{n_1} \ldots \mathrm{tr}\, T_p^{n_v} = O(p^v) = o(p^{k-w+1})$$

Therefore, in this case

$$S_{A^*,B} = o(p^{k-w+1}) .$$

Case 2. Some B class contains more than 2 elements. By Lemma 3 and the inclusion-exclusion principle, it is seen in the same manner that

$$S_{A^*,B} = o(p^{k-w+1}).$$

Case 3. B is a pairing and $G(A^*,B)$ has $k-w+1$ cycles. combining the above discussion

$$E\,M_k = \frac{1}{pm^k} \sum_{A(j)|A} \sum_{\substack{B \geq A^* \\ B \text{ even}}} \sum_{(i)|A^*,B} K(A^*,B)\, E\, t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1}$$

$$= \frac{1}{pm^k} \sum_A m(m-1)\ldots(m-w+1)\left(\sum{}' + \sum{}'' + \sum{}'''\right) \sum_{(i)|A^*,B} K(A^*,B) E t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1} .$$

Here, w is the number of classes of A, $\sum'$, $\sum''$, $\sum'''$ correspond Case 1, Case 2 and Case 3 respectively. Notice that for Case 3, $K(A,B) = 1$. Thus

$$E\,M_k = \sum_{w=1}^{k} \sum_{\substack{A, A \text{ has} \\ w \text{ classes}}} \frac{m^w}{pm^k} \sum_{\substack{B > A^* \\ B \text{ even}}}{}''' \sum_{(i)|A^*,B} E(t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1}) + o(1).$$

We need the following lemma.

Lemma. Given a partition A of $\{1,2,\ldots,k\}$ with w classes and then a partition A^* of $\{1,2,\ldots,2k\}$, there is at most one pairing B subject to A^* such that $G(A^*,B)$ is a Q-graph with $k-w+1$ cycles.

Proof. Suppose $G(A^*,B)$ has $k-w+1$ cycles. Then,

1^o. If i_{2r} and i_{2r+1} belong to the same A^* class, they must be identified. Otherwise, there would be a cycle meets two vertices with a class.

2^o. There are no sequences of the form:

$$A^*_{a_1},\ L_1,\ A^*_{a_2},\ L_2,\ A^*_{a_3},\ldots,A^*_{a_r},\ L_r,\ A^*_{a_1},$$

here $A^*_{a_1}, A^*_{a_2},\ldots,A^*_{a_r}$ are different classes and L_q is a simple path begins at a vertex in $A^*_{a_q}$ and ends at a vertex in $A^*_{a_{q+1}}$ $(q = 1,\ldots,r;\ a_{r+1} = a_1)$, and the end of L_{q-1} and the beginning of L_q are not identified though they are both in $A^*_{a_q}$.

For, if such sequence exists, L_q should be completed by other path into a cycle C_q, $q = 1,\ldots,r$. If these cycles are different, then we will have a sequence

$$A^*_{a_1},\ C_1,\ A^*_{a_2},\ C_2,\ldots,A^*_{a_r},\ C_r,\ A^*_{a_1}$$

which is prohibited by Lemma 1. Suppose $C_{d+1} = C_1$ and $C_1,\ldots,C_d$ are distinct, we would have the prohibited sequence

$$A^*_{a_2},\ C_2,\ A^*_{a_3},\ldots,C_d,\ A^*_{a_{d+1}},\ C_1,\ A^*_{a_2}.$$

Thus 2^o is proved.

If $G(A^*,B)$ has $k-w+1$ cycles, then as we just proved, 1^o, 2^o are true. And if we start from $i_2 i_3$, preserving 1^o and 2^o, we see that it is determined completely which indices should be identified. Thus $G(A^*,B)$ is unique.

By applying this lemma,

$$E M_k = \sum_{w=1}^{k} \sum_{A}{}' \frac{m^w}{pm^k} \sum_{(i)\,|\,A^*,B} E(t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1}) + o(1).$$

Here, $\sum_A{}'$ means summation over those partition A of $\{1,2,\ldots,k\}$ with w classes, for which there is a pairing B of $\{1,2,\ldots,2k\}$, $B \geq A^*$, such that the Q-graph $G(A^*,B)$ has just $k-w+1$ cycles.

If among the $k-w+1$ cycles of $G(A^*,B)$, there are n_1 loops, n_2 cycles with 2 vertices,..., n_w cycles with w vertices, then

$$n_1 + n_2 + \ldots + n_w = k-w+1,$$
$$n_1 + 2n_2 + \ldots + wn_w = k,$$

and by Lemma 3 and inclusion-exclusion principle

$$\sum_{(i)|A^*,B} t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1} = (\mathrm{tr}T_p)^{n_1} (\mathrm{tr}T_p^2)^{n_2} \cdots (\mathrm{tr}T_p^w)^{n_w} + o(p^{k-w+1}).$$

But by Lemma A, for all $r \geq 1$, as $p \to \infty$

$$E\left(\frac{1}{p}\mathrm{tr}T_p\right)^{n_1} \left(\frac{1}{p}\mathrm{tr}T_p^2\right)^{n_2} \cdots \left(\frac{1}{p}\mathrm{tr}T_p^w\right)^{n_w} \longmapsto H_1^{n_1} H_2^{n_2} \cdots H_w^{n_w}$$

we have

$$E_k = \lim_{p\to\infty} E\, M_k = \sum_{w=1}^{k} y^{k-w} \sum_{\substack{n_1+n_2+\ldots+n_w=k-w+1 \\ n_1+2n_2+\ldots+wn_w=k}} H_1^{n_1} H_2^{n_2} \cdots H_w^{n_w}\, N_{n_1 n_2 \ldots n_w}.$$

Here $N_{n_1 n_2 \ldots n_w}$ is the number of those partitions A of the set $\{1,2,\ldots k\}$ into w classes, for each such A there exists a pairing B of $\{1,2,\ldots,2k\}$, $B \geq A^*$, and the Q-graph $G(A^*,B)$ has n_1 loops, n_2 cycles with 2 vertices..., n_w cycles of w vertices.

For evaluating $N_{n_1 n_2 \ldots n_w}$, we note that to each such $G(A^*,B)$ we can construct a finite sequence of integers in the following way: we draw this graph along the order $i_2 i_3, i_4 i_5, \ldots, i_{2k} i_1$, and the sequence is defined as

1^o. The 1st term is 0,

2^o. The 2nd, 4-th,...,2k-th terms are 1,

3^o. If $i_{2r} i_{2r+1}$ just completes a cycle of length s, then the 2r+1-th term is -s, otherwise 2r+1-th term is 0.

Such sequence has the properties that: it has 2k+1 terms, even number terms are 1, odd number terms ≤ 0, total sum is 0, partial sums ≥ 0.

It is easy to see that $N_{n_1 \ldots n_w}$ is the number of such sequences, in which there are n_1 places with -1, n_2 places with $-2, \ldots, n_w$ places with $-w$. With the aid of a lemma in Jonsson [4,5], it is seen that

$$N_{n_1 \ldots n_w} = \frac{1}{k+1} \frac{(k+1)!}{n_1 \ldots n_w (k+1-(k-w+1))!}$$

$$= \frac{k!}{n_1! \ldots n_w!\ w!}$$

Then

$$E_k = \sum_{w=1}^{k} y^{k-w} \sum_{\substack{n_1+\ldots+n_w=k-w+1 \\ n_1+\ldots+wn_w=k}} \frac{k!}{n_1! \ldots n_w!\ w!} H_1^{n_1} \ldots H_w^{n_w}.$$

By a well-known inequality about moments, if k is even,

$$|H_1^{n_1} H_2^{n_2} \ldots H_w^{n_w}| \leq H_k^{\frac{1}{k}(n_1+2n_2+\ldots+wn_w)} = H_k$$

Thus

$$|E_k| \leq H_k \sum_{w=1}^{k} y^{-w} \frac{k!}{(k-w+1)!w!} \sum_{\substack{n_1+\ldots+n_w=k-w+1 \\ n_1+\ldots+wn_w=k}} \frac{(k-w+1)!}{n_1! \ldots n_w!} y^{n_1} y^{n_2} \ldots y^{wn_w}$$

The inner sum is the y^k term of the polynomial $(y+y^2+\dots+y^w)^{k-w+1}$. But this polynomial is dominated by the power series

$$(y+y^2+\dots)^{k-w+1} = y^{k-w+1}(1-y)^{-k+w-1}$$

its y^k term is

$$\frac{(k-w+1)(k-w+2)\dots(k-1)}{(w-1)!}\,y^k = \frac{(k-1)!}{(w-1)!(k-w)!}\,y^k.$$

Therefore, if k is even,

$$|E_k| \le H_k \sum_{w=1}^{k} y^{-w} \frac{k!}{(k-w+1)!w!}\,\frac{(k-1)!}{(w-1!(k-w)!}\,y^k$$

$$= H_k \sum_{v=0}^{k-1} y^v \frac{k!}{(v+1)!(k-v)!}\,\frac{(k-1)!}{(k-v-1)!v!}$$

$$= H_k \sum_{v=0}^{k-1} y^v \binom{k}{v}\binom{k}{v+1}\frac{1}{k} \le H_k(1+y)^{2k}.$$

So,

$$\sum_{k\ \text{even}} |E_k|^{-\frac{1}{k}} \ge \frac{1}{(1+y)^2} \sum_{k\ \text{even}} H_k^{-\frac{1}{k}} = +\infty.$$

4. ASYMPTOTIC LIMIT OF VARIANCE OF M_k

In this section we prove that for any integer $k \geq 1$

$$\text{Var } M_k = EM_k^2 - (EM_k)^2 \to 0 \quad \text{as } p \to \infty.$$

Then we will complete the proo: of the theorem.

We have

$$EM_k^2 = \frac{1}{p^2 m^{2k}} E\Bigg[\sum_A \sum_{(j)|A} \sum_{(i)} \sum_{A'} \sum_{(j')|A'} \sum_{(i')} \prod_{q=1}^{k} \left(X_{i_{2q-1}j_q} X_{i_{2q}j_q}\right)$$

$$\prod_{q'=1}^{k} (X_{i'_{2q'-1}j'_{q'}} X_{i'_{2q'}j'_{q'}}) t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1} t_{i'_2 i'_3} t_{i'_4 i'_5} \cdots t_{i'_{2k} i'_1}\Bigg]$$

Here,

A —— a partition of $\{1,2,\ldots,k\}$,

A'—— a partition of $\{k+1,\ldots,2k\}$,

$(j) = (j_1,\ldots,j_k)$, $1 \leq j_1 \leq m,\ldots, 1 \leq j_k \leq m$,

$(j') = (j'_1,\ldots,j'_k) = (j_{k+1},\ldots,j_{2k})$, $1 \leq j'_1 \leq m,\ldots, 1 \leq j'_k \leq m$,

$(i) = (i_1,\ldots,i_{2k})$, $1 \leq i_1 \leq p,\ldots, 1 \leq i_{2k} \leq p$,

$(i') = (i'_1,\ldots,i'_{2k}) = (i_{2k+1},\ldots,i_{4k})$, $1 \leq i'_1 \leq p,\ldots, 1 \leq i'_{2k} \leq p$,

$(j)|A$ means $j_a = j_b$ iff a and b belong to the same A class.

$(j')|A'$ means $j'_a = j'_b$ iff a and b belong to the same A' class.

We split the sum for EM_k^2 into two parts:

$$EM_k^2 = \frac{1}{p^2 m^{2k}} \sum\nolimits_1 + \frac{1}{p^2 m^{2k}} \sum\nolimits_2.$$

In $\sum_1$ we collect all those terms in which some coordinates of (j) equal some coordinates of (j'). In $\sum_2$ we collect all other terms.

But $\sum_2 = \sum_3 - \sum_4$, here

$$\sum_3 = E\left[\sum_A \sum_{(j)|A} \sum_{(i)} \left\{E \prod_{q=1}^{k} \left(X_{i_{2q-1}j_q} X_{i_{2q}j_q}\right)\right\} t_{i_2i_3} t_{i_4i_5} \cdots t_{i_{2k}i_1} \cdot \right.$$

$$\left. \sum_{A'} \sum_{(j')|A'} \sum_{(i')} \left\{E \prod_{q=1}^{k} \left(X_{i'_{2q-1}j'_q} X_{i'_{2q}j'_q}\right)\right\} t_{i'_2i'_3} t_{i'_4i'_5} \cdots t_{i'_{2k}i'_1}\right]$$

$$\sum_4 = E\left[\sum_A \sum_{(j)|A} \sum_{(i)} \left\{E \prod_{q=1}^{k} \left[X_{i_{2q-1}j_q} X_{i_{2q}j_q}\right]\right\} t_{i_2i_3} t_{i_4i_5} \cdots t_{i_{2k}i_1} \cdot \right.$$

$$\left. \sum_{A'} \sum_{(j')|A', (j')\cap(j)\neq\phi} \sum_{(i')} \left\{E \prod_{q=1}^{k} \left(X_{i'_{2q-1}j'_q} X_{i'_{2q}j'_q}\right)\right\} t_{i'_2i'_3} t_{i'_4i'_5} \cdots t_{i'_{2k}i'_1}\right]$$

From Section 3 and the hypothesis of the theorem, it is seen that

$$\frac{1}{p^2 m^{2k}} \sum_3 \to \left(\lim_{p\to\infty} EM_k\right)^2 = E_k^2 .$$

In $\sum_4$, $(j')|A'$ has w' free indices, here w' is the number of classes in A'. But $(j')\cap(j)\neq\phi$, so some j'_a have to be fixed to some j_b. Thus the number v' of free indices in $(j')|A'$ is less than w'. So,

$$\sum_{A'} \sum_{(j')|A', (j')\cap(j)\neq\phi} \sum_{(i')} = \sum_{A'} O(m^{w'-1} p^{k-w'+1}) = O(p^k),$$

and as $p \to \infty$,

$$\frac{1}{p^2 m^{2k}} \sum_4 = \frac{1}{p^2 m^{2k}} E\left[\sum_A m^w O(p^{k-w+1}) O(p^k)\right] = O(\tfrac{1}{p}) \to 0.$$

Therefore, as $p \to \infty$,

$$\frac{1}{p^2 m^{2k}} \sum_2 \to E_k^2 .$$

As for $\sum_1$, we at first fix A and A'. Under $(j)|A$, the k indices $j_1,\ldots,j_k$ reduce to w different values $h_1,\ldots,h_w$, and under $(j')|A'$, the k indices $j_1',\ldots,j_k'$ reduce to w' different values $h_1',\ldots,h_{w'}'$. Now in $\sum_1$, some h_a' must equal some h_b. As an example, suppose $h_1' = h_1$, $h_2' = h_2$, but no other such relations. In this case, we consider the partition of $\{1,2,\ldots,2k\}$:

$$\tilde{A} = (\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_{w+w'-2}) = (A_1\cup A_1',\ A_2\cup A_2',\ A_3,\ldots,A_w,\ A_3',\ldots,A_{w'}').$$

In order that a term in $\sum_1$ corresponding A,A' and $h_1 = h_1'$, $h_2 = h_2'$, does not equal 0, i.e.

$$E\left(\prod_{q=1}^{2k} X_{i_{2q-1}j_q} X_{i_{2q}j_q}\right) E\left(t_{i_2i_3}t_{i_4i_5}\cdots t_{i_{2k}i_1}\cdot t_{i_{2k+2}i_{2k+3}}t_{i_{2k+4}i_{2k+5}}\cdots t_{i_{4k}i_{2k+1}}\right) \neq 0,$$

it is necessary that there is a pairing B of $\{1,2,\ldots,4k\}$ such that each pair is included in an A^* class, here the * has the same meaning as before, and if a,b is such a pair then $i_a = i_b$. Thus, under such identifications, $i_2i_3, i_4i_5,\ldots i_{2k}i_1,\ i_{2k+2}i_{2k+3},\ i_{2k+4}i_{2k+5},\ldots i_{4k}i_{2k+1}$ constitute a multi-graph, consisting of disjoint cycles. This graph is a Q-graph, because it is easy to see that the $w+w'-2$ classes are connected by these edges. So it has at most $2k-w-w'+2+1$ cycles. If there need more identifications, we can quotate Lemma 3. Then, as $p\to\infty$,

$$\frac{1}{p^2m^{2k}}\sum_1 \doteq \frac{1}{p^2m^{2k}}\sum_A\sum_{A'} O\left(m^{w+w'-r}p^{2k-w-w'+r+1}\right) = O(p^{-1}) \to 0,$$

here r is the general number instead of the specific 2 in the example.

Therefore

$$\mathrm{Var}\ M_k = EM_k^2 - (EM_k)^2 \to E_k^2 - E_k^2 = 0.$$

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In this paper, the authors showed that the spectral distribution of a sequence of the products of random matrices will tend to a distribution function in the limit as the number of variables tend to infinity.

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